MATH 2050C Lecture on 2/26/2020

ε given first find K depending on ε

Recall: $\boxed{\lim(x_n) = x}$ iff $\forall \varepsilon > 0,\ \exists K = K(\varepsilon) \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq K$

- (x_n) is convergent if $\lim(x_n) = x$ for some $x \in \mathbb{R}$

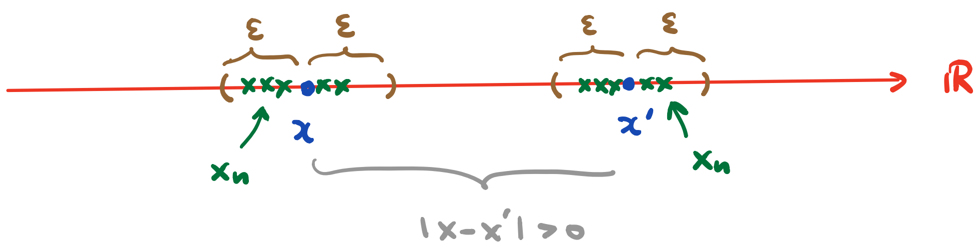
maybe unknown beforehand

- otherwise, (x_n) is divergent.

Prop: Any convergent seq. (x_n) has a unique limit.

Proof: (By contradiction) Suppose $\lim(x_n) = x$ and $\lim(x_n) = x'$ where $x \neq x'$.

Since $x \neq x'$, we can take $\boxed{\varepsilon := \dfrac{|x - x'|}{4} > 0}$.



- $(x_n) \to x \Rightarrow$ for this particular $\varepsilon > 0$, $\exists K = K(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x| < \varepsilon \qquad \forall n \geq K.$$

- $(x_n) \to x' \Rightarrow$ for this particular $\varepsilon > 0$, $\exists K' = K'(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x'| < \varepsilon \qquad \forall n \geq K'$$

Take $\bar{K} := \max\{K, K'\} \in \mathbb{N}$, then

$$\begin{aligned} |x - x'| &= |(x_{\bar{K}} - x) - (x_{\bar{K}} - x')| \\ &\leq |x_{\bar{K}} - x| + |x_{\bar{K}} - x'| \\ &< \varepsilon + \varepsilon = 2\varepsilon = \frac{|x - x'|}{2} \end{aligned}$$

So, $0 < |x - x'| < \dfrac{|x - x'|}{2}$ Contradiction!

$\square$

To Show $\lim(x_n) = x$

- General strategy: Find (a_n) s.t. $|x_n - x| \leq a_n$ for n large

  and $\lim(a_n) = 0$

  simplify / estimate

  Want: $a_n < \varepsilon$

- Useful trick: $\boxed{\dfrac{\text{Smaller}}{\text{Bigger}} \leq \dfrac{\square}{\square} \leq \dfrac{\text{Bigger}}{\text{Smaller}}}$

More Examples

(a) $\boxed{\lim\left(\dfrac{1}{1+na}\right) = 0}$ for any fixed $a > 0$

Pf: Let $\varepsilon > 0$ be fixed.

Choose $K = K(\varepsilon) \in \mathbb{N}$ s.t. $K > \dfrac{1}{\varepsilon a}$ (>0)

$\forall\, n \geq K$, we have

$$\left|\frac{1}{1+na} - 0\right| = \frac{1}{1+na} < \frac{1}{na} \leq \frac{1}{Ka} < \varepsilon$$

$$\left|\frac{1}{1+na} - 0\right| = \frac{1}{1+na} < \frac{1}{na} \overset{?}{<} \varepsilon$$

$$\Uparrow$$

$$\frac{1}{\varepsilon a} < n$$

$\square$

(b) $\boxed{\lim(b^n) = 0}$ provided that $b \in (0,1)$.

Pf: Since $b \in (0,1)$, we can write $b = \dfrac{1}{1+a}$ for some $a > 0$.

$$\lim(b^n) = 0 \iff \lim\left(\frac{1}{(1+a)^n}\right) = 0$$

Observe that [Recall: $(1+x)^n \geq 1 + nx \quad \forall\, x > -1$]

$$\left|\frac{1}{(1+a)^n} - 0\right| = \frac{1}{(1+a)^n} \leq \underbrace{\frac{1}{1+na}}$$

$\to 0$ by Example (a)

Ex: Complete the proof.

$\square$

(c) $\boxed{\lim (c^{1/n}) = 1}$ where $c > 0$ is fixed.

Pf: 3 cases: $c = 1$, $c > 1$, $c < 1$.

Case 1: $c = 1 \Rightarrow c^{1/n} = 1 \quad \forall n \in \mathbb{N}$ Trivial.

Case 2: $c > 1$.

Since $c^{1/n} > 1 \quad \forall n \in \mathbb{N}$ (Pf: By M.I.),

we can write for each $n \in \mathbb{N}$,

$$c^{1/n} = 1 + d_n \quad \text{where } d_n > 0.$$

$$\Rightarrow \quad c = (1+d_n)^n \underset{\substack{\uparrow \\ \text{Bernoulli}}}{\geq} 1 + n d_n$$

$$\Rightarrow \quad d_n \leq \frac{c-1}{n} \to 0 \text{ as } n \to \infty$$

$|c^{1/n} - 1|$

$= |(1+d_n) - 1|$

$= |d_n| = d_n < \varepsilon$

Let $\varepsilon > 0$. Choose $K \in \mathbb{N}$ s.t. $K > \frac{c-1}{\varepsilon}$ (>0)

$\forall n \geq K$, we have

$$\left| c^{1/n} - 1 \right| = |d_n| = d_n \leq \frac{c-1}{n} \leq \frac{c-1}{K} < \varepsilon$$

Case 3: $0 < c < 1$.

"Sketch": $1 > c^{1/n} = \frac{1}{1+h_n}$ for some $h_n > 0$.

$$\left| c^{1/n} - 1 \right| = \left| \frac{1}{1+h_n} - 1 \right| = \left| \frac{h_n}{1+h_n} \right| \leq h_n \leq \frac{1}{nc} \to 0 \text{ as } n \to \infty.$$

Since $c^{1/n} = \frac{1}{1+h_n}$, raise to n^{th} power.

Ex: Complete this proof.

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} \leq \frac{1}{nh_n}$$

$$\Rightarrow \quad h_n \leq \frac{1}{nc}$$